Geodesic Math

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excerpt of an article by Joe Clinton

Producing geodesics from the icosahedron.

The following article is an excerpt of an article by Joe Clinton on the different methods of producing geodesics from the icosahedron. Formatted by Jay Salsburg, Design Scientist

Using analytical geometry (Fuller used spherical trigonometry), calculated on a computer:

General procedure

1. find the 3-dimensional coordinates of the vertices of the grid on the spherical surface
2. find geometry using the different **Methods**
3. calculate the chord lengths, angles etc. with these coordinates and analytical formulas.

Joe worked with Fuller on his programs and was funded by NASA on a project called “Structural Design Concepts for Future Space Missions.”

The specific motivation for developing these methods was to have a variety of forms to combine in large space frame domes. For example the Expo dome in Montreal is a combination of a:

32-frequency regular triacon (Class II, method 3) and a

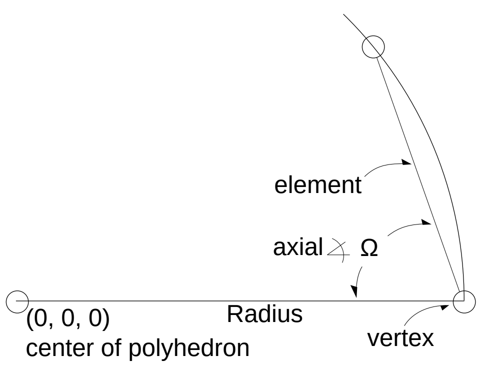
16-frequency truncatable alternate (Class I, method 3)

With known parameters and sophisticated analysis, large structures can be optimized by different combinations and different methods; however, for small structures (up to 40') they are not generally relevant. What was called “alternate” breakdown, Joe classifies as “Class I’’; what was called ‘’triacon’’ he classifies as ‘’Class II’’. Joe wrote this section mainly with the intent of communicating the state of development of geodesic geometries and the hope that it would be an aid to those interested in exploring and expanding this field.

Geodesic Math

DEFINITIONS

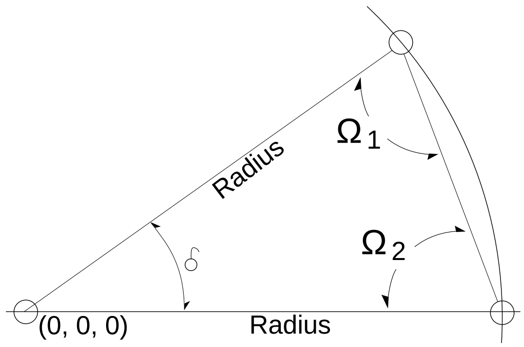
Axial angle (omega Ω) = an angle formed by an element and a radius from the center of the polyhedron meeting in a common point. The vertex of the axial angle is chosen as that point common to the polyhedron element and radius.



The axial angle Ω may be found if the central angle δ is known by the following equation:

$$\Omega = \frac{180 - \delta}{2}$$

Central angle (delta , δ) = an angle formed by two radii of the polyhedron passing through the end points of an element of the polyhedron. The vertex of the central angle is chosen as that point common to both radii (the center of the polyhedron).



The central angle δ may be found by knowing the axial angles Ω_1,& Ω_2 at each end of an element.

$$\delta = 180 - (\Omega_1 + \Omega_2)$$

Chord factor (cf) = the element lengths calculated based on a radius of a non-dimensional unit of one for the spherical form with the end points of the elements coincident with the surface of the sphere.

If the central angle δ is known, the chord factor may be calculated as follows:

$$cf = 2 \sin\frac{\delta}{2}$$

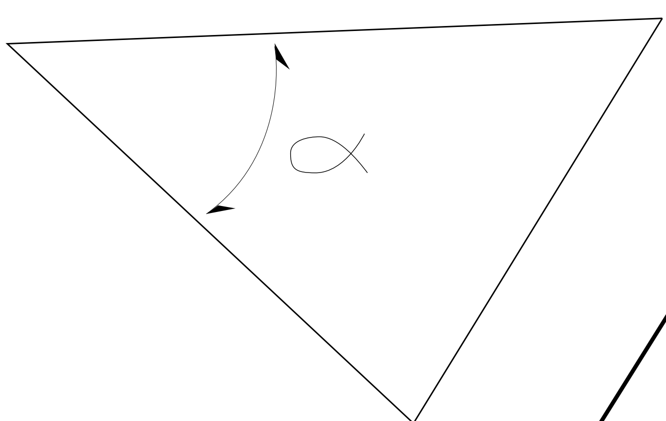
The length of any element for larger structures may be found by the equation:

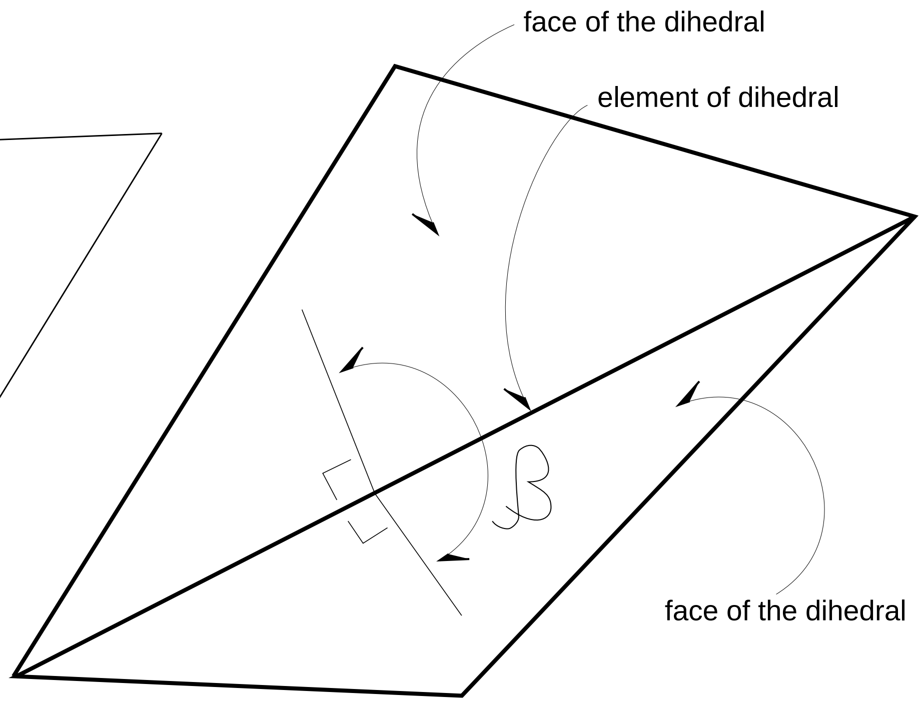
I = cf x r

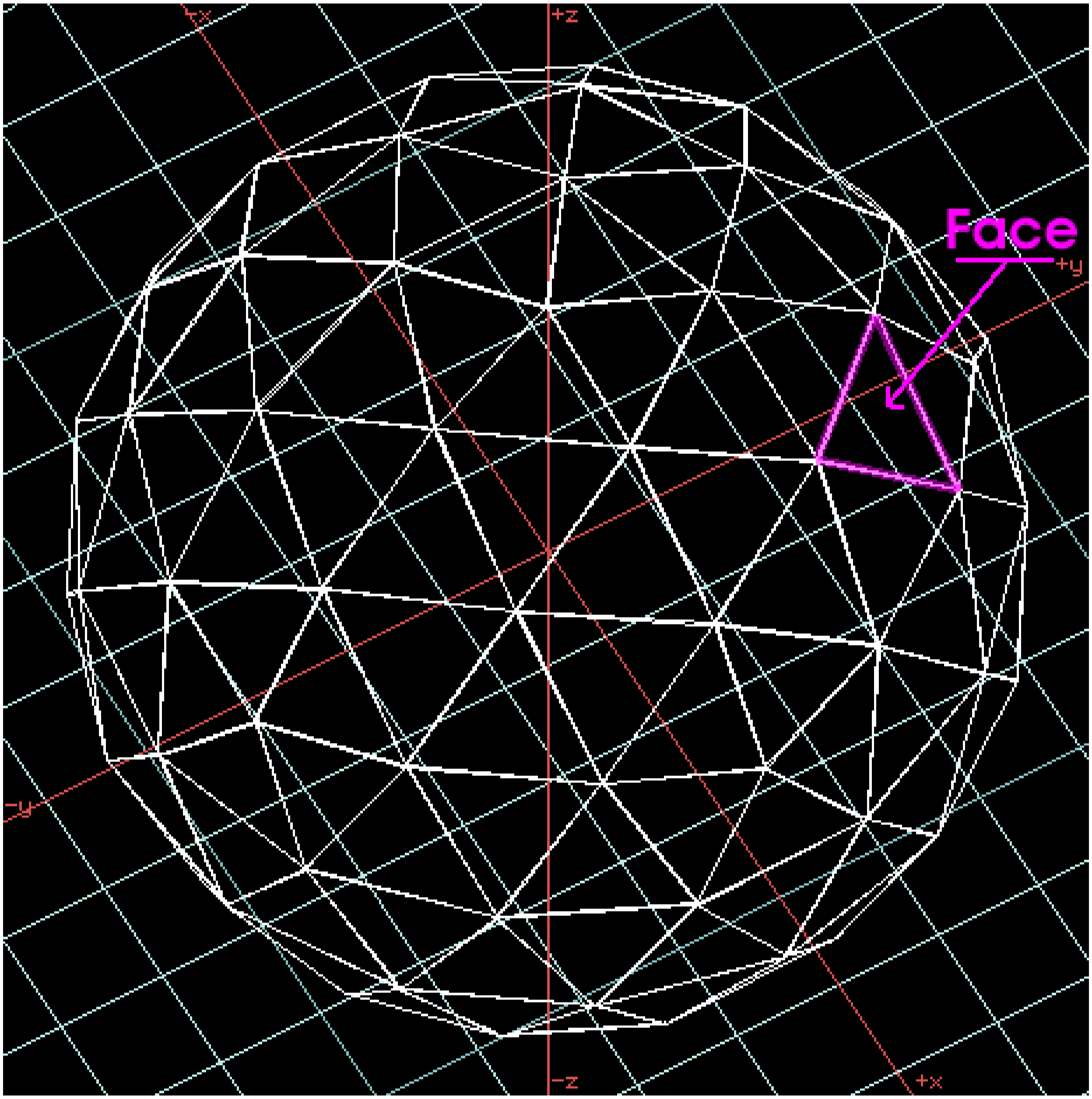
where: r = the radius of the desired structural form
I = the length of the new element

Dihedral angle (beta β) = an angle formed by two planes meeting in a common line. The two planes themselves are faces of the dihedral angle, and the element is the common line. To measure the dihedral angle measure the angle whose vertex is on the element of the dihedral angle and whose sides are perpendicular to the element and lie one in each face of the dihedral angle.

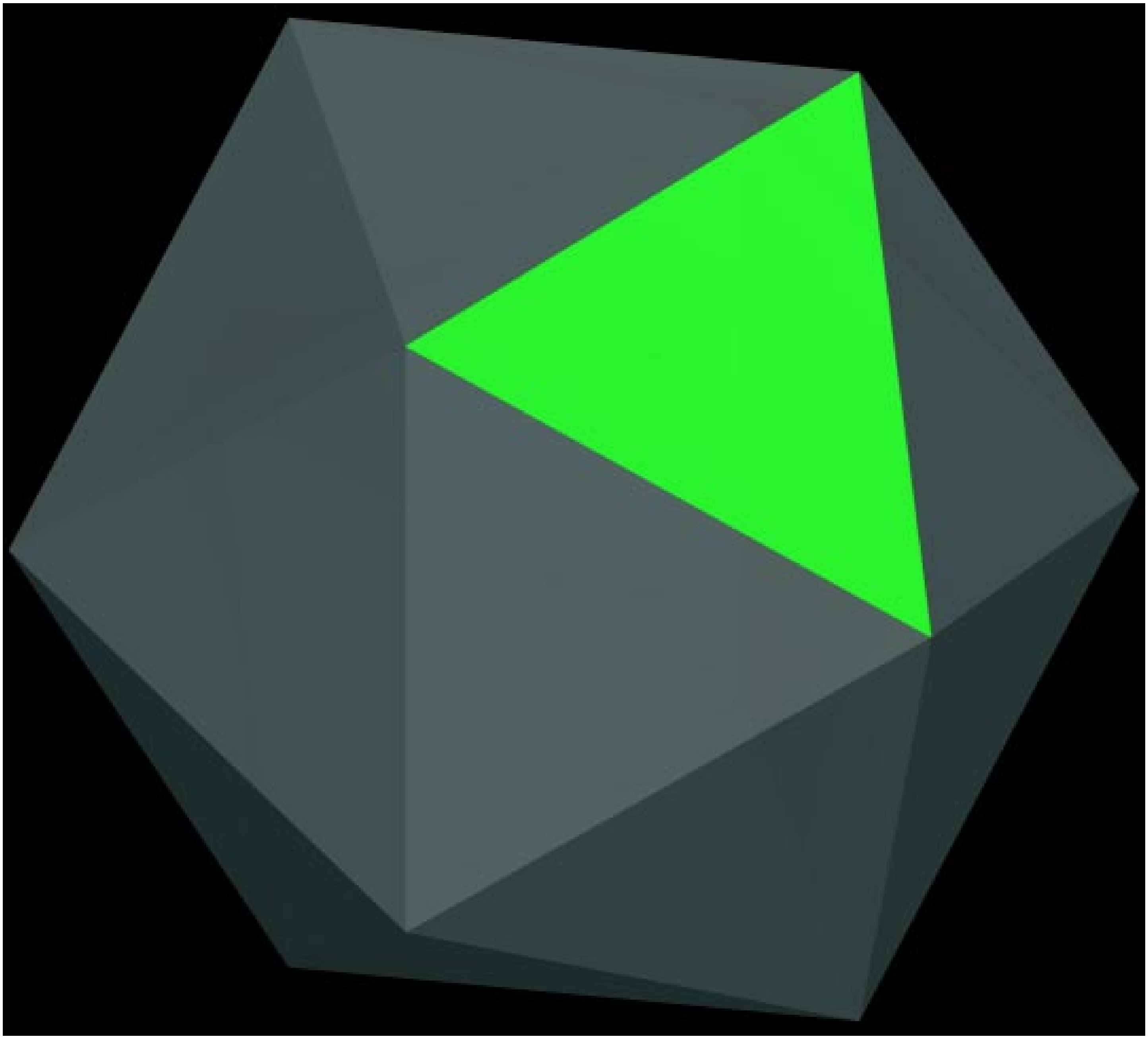
Face angle (alpha α) = an angle formed by two elements meeting in a common point and lying in a plane that is one of the faces of the polyhedron.



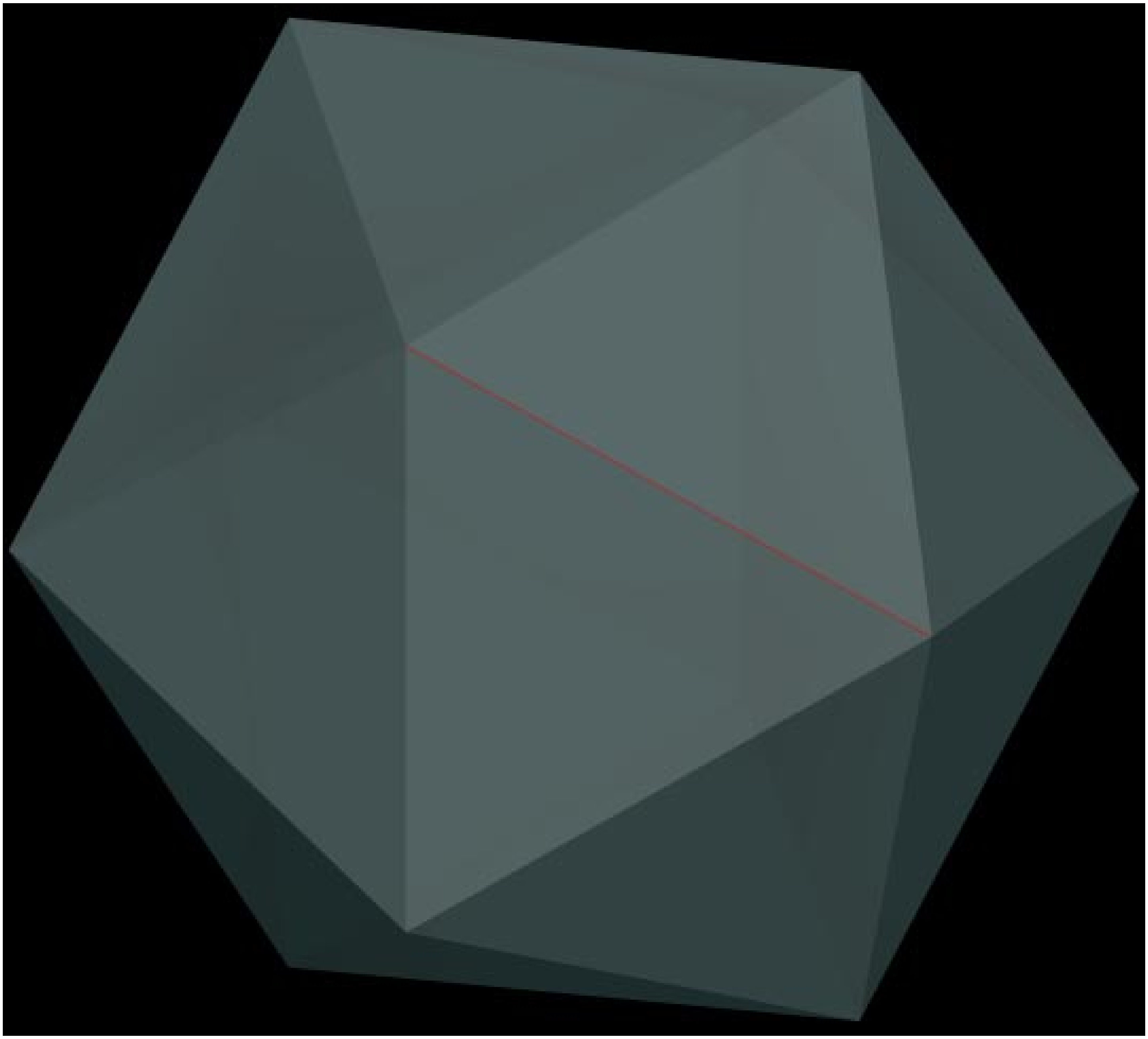




Face = any of the plane polygons making up the surface of the structural form.

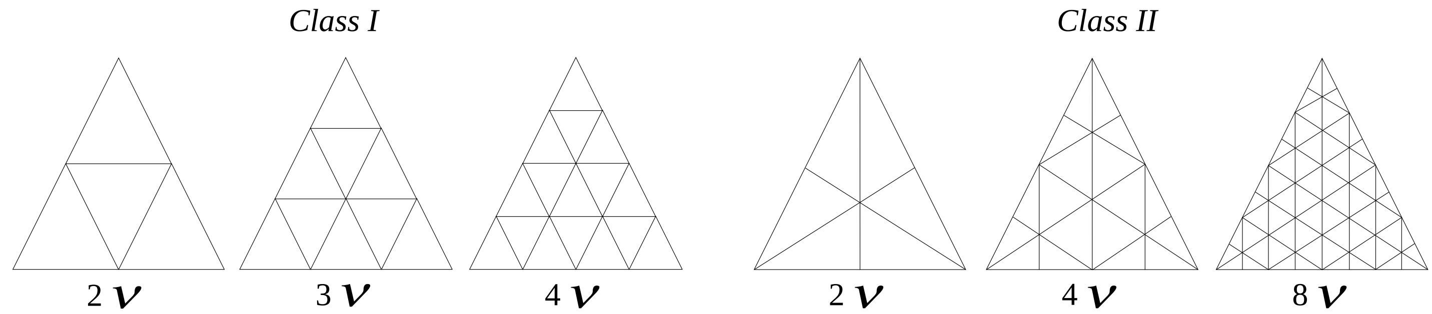


Principle polyhedral triangle (PPT) = any one of the plane equilateral triangles which form the faces of the regular polyhedron.

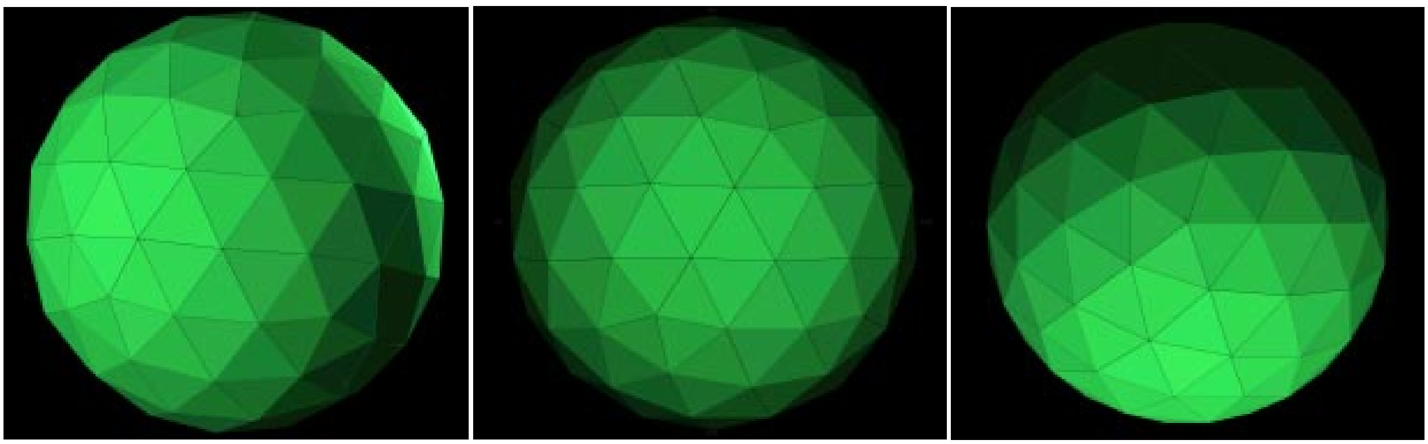


Principle side (PS) = any one of the sides of the principle polyhedral triangle.

Frequency (Nu ν) = the number of parts or segments into which a principle side is subdivided.



Orientation = the orientation that the Polyhedral form has in space with respect to the observer are named:



face **edge** **vertex**

METHODS OF GENERATING 3-WAY GEODESIC GRIDS

Upon using the spherical form as a structural unit, it is apparent that the basic polyhedral form, in its fundamental state, can not satisfy the range of conditions that must be geometrically and structurally met. There have been many methods developed for reducing the basic polyhedral form into a larger number of components from which the geometrical properties may be made to remain within the structural fabrication and erection limits for a desired configuration.

Several methods of generating 3-way geodesic grids are discussed here in a broad sense to give the experimenter a basis from which other methods may be developed.

The methods described here may be considered as having characteristics of one of the two following classifications:

Class 1 or alternate

- based on regular polyhedral forms, most generally the icosahedron.
- frequency of subdivision may be odd or even.

$$V = 10\nu^2 + 2$$

$$F = 20\nu^2$$

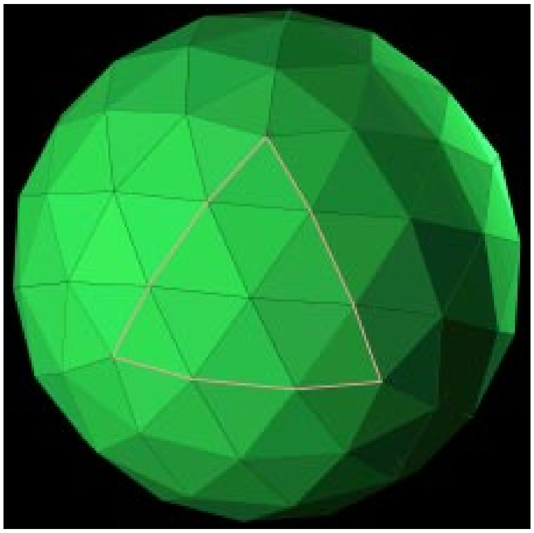
$$E = 30\nu^2$$

where:

V = the number of vertices
F = the number of faces
E = the number of edges
} for total icosahedral sphere

ν = frequency of subdivision

-demonstrates symmetries as illustrated here:



Class II or traicon

-based on the quasi-regular polyhedral forms, most generally the rhombic triacontahedron.
-frequency of subdivision may only be even.

$$V = \eta + 2$$

$$F = 2(\eta)$$

$$E = 3(\eta)$$

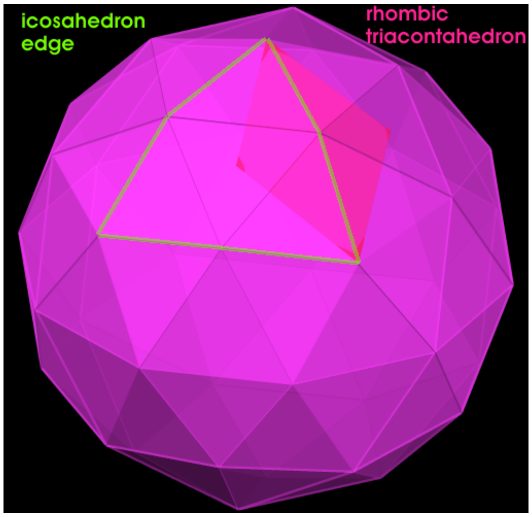
where:

$$\eta = \frac{15\nu^2}{2}$$

V = the number of vertices
F = the number of faces
E = the number of edges
} for total rhombic triacontahedral sphere

ν = frequency of subdivision

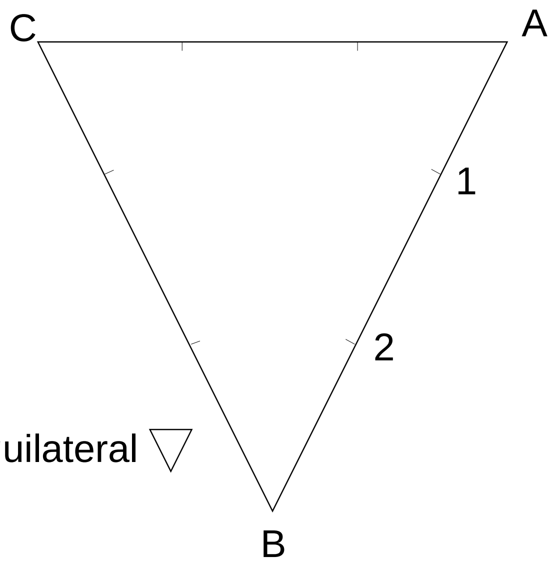
- demonstrates symmetries as illustrated here:



The green line is the edge of the icosahedron. The submerged red diamond object is the “triacon” face. It is beneath the surface in the graphic shown here because the edges are not coincident with the icosahedron faces, they are partially devised from the face centers of the icosa. This renders the “triacon” under the icosa faces. Due to the symmetrical characteristics of the basic polyhedral form only one face, or portions of one face, of the polyhedron is used for calculating the geometrical properties of the structural configuration. The remaining faces may be found by rotations and/or reflections of this principle polyhedral triangle and its transformations.

CLASS I

Method 1 or ‘’alternate’’



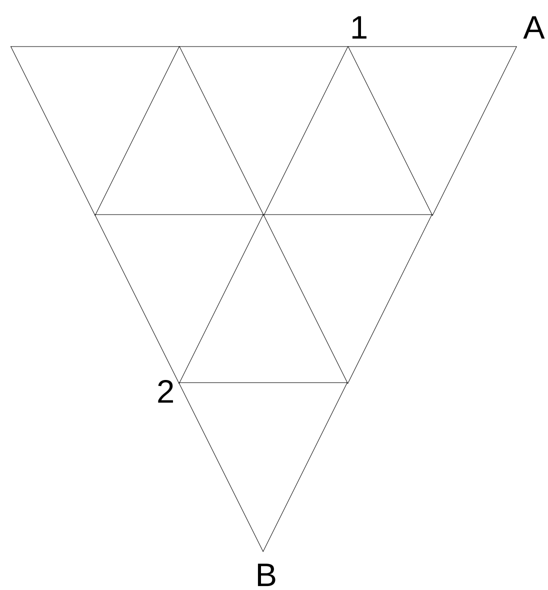
Note: $\overline{A1} = \overline{12}$

▽ ABC is a plane equilateral ▽

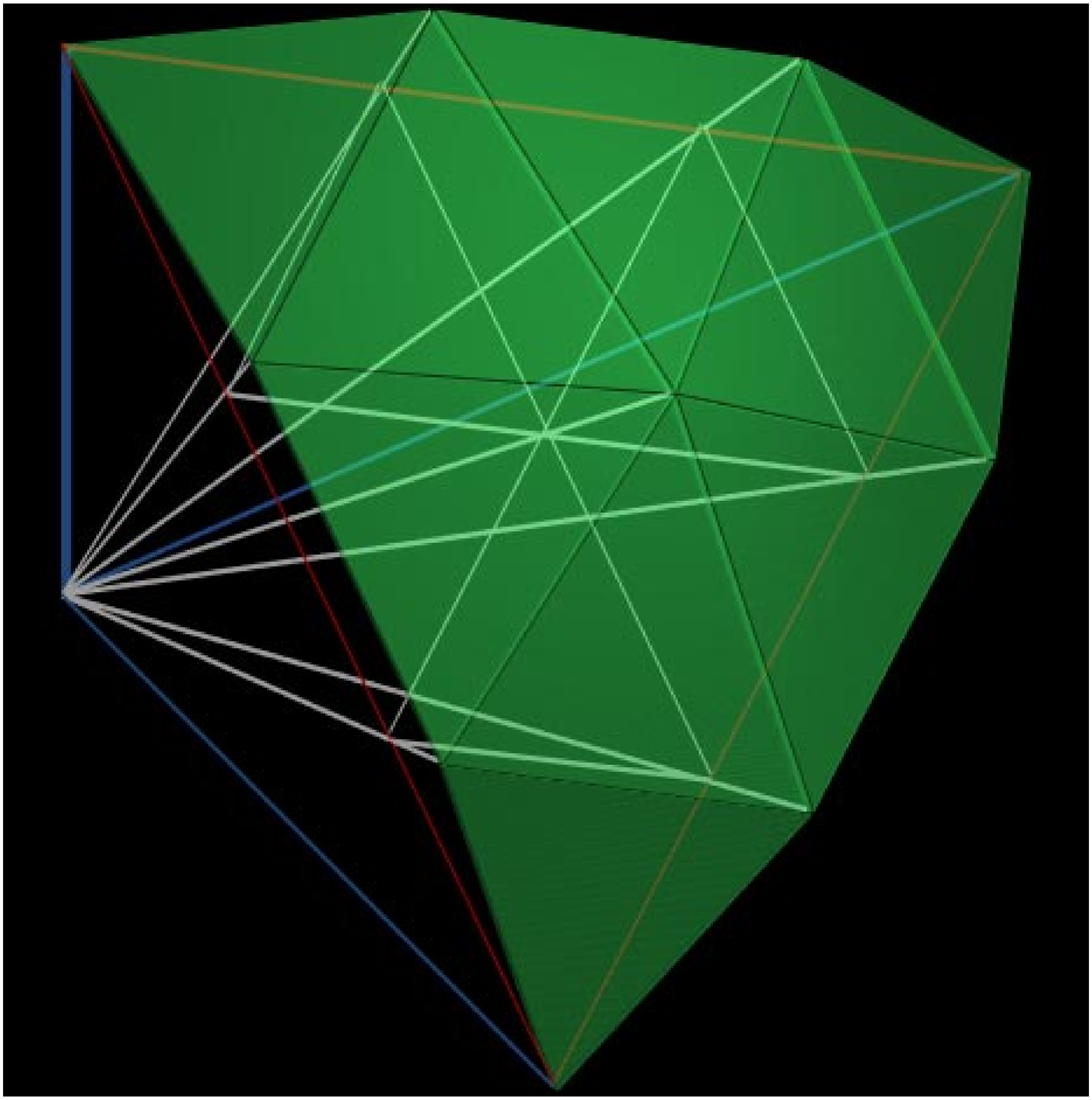
The PPT is subdivided into n frequency, with the parts chosen as equal divisions along the three principle sides.

Each point of subdivision is then connected with a line segment parallel to their respective sides thereby giving a 3-way grid so that a series of equilateral triangles are formed.

NOTE: AB is parallel to 12

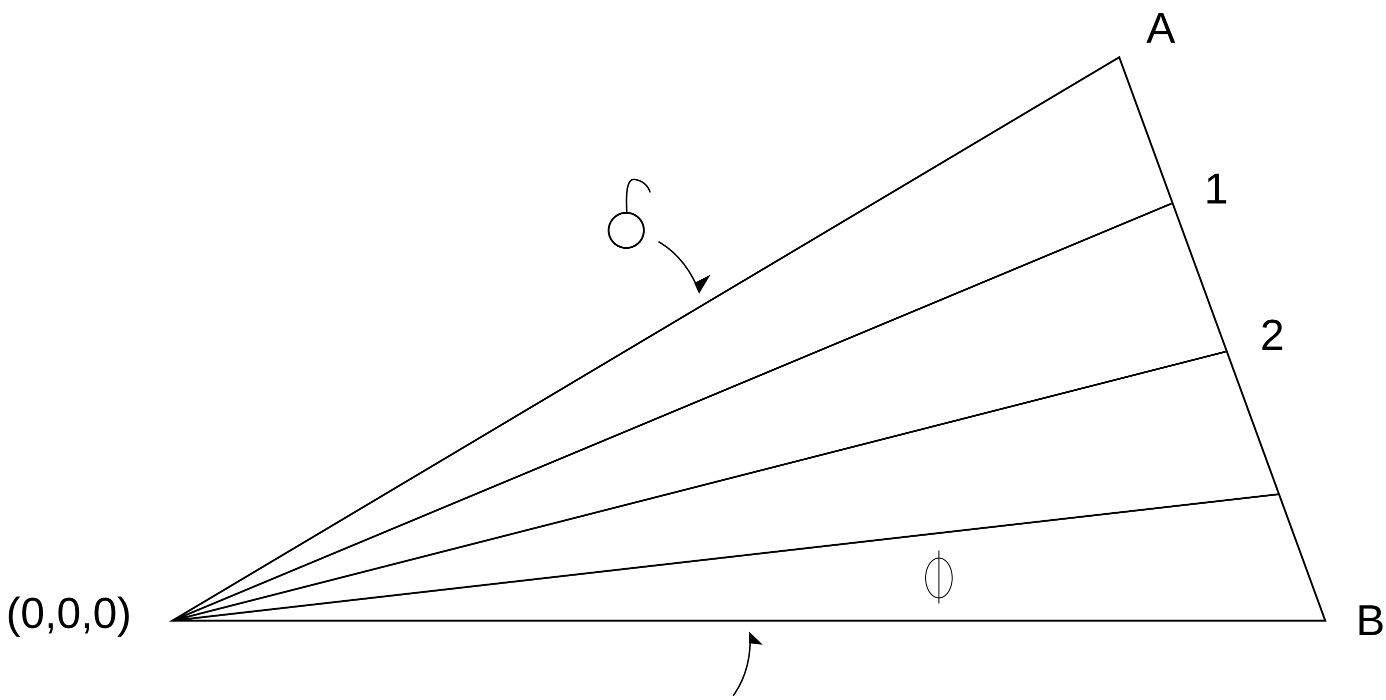


Each vertex on the PPT is then translated along a line passing through the origin (0, 0, 0) of the polyhedron and its respective vertex, onto the surface of the circumscribed sphere. The element connecting the translated vertices form the chords of a 3-way great circular grid.



Method 2: (This method produces equal divisions along the spherical PPT and results in, for example, 3 different triangles in the 3 ν with 3 different strut lengths. Method I has 2 different triangles and 3 different struts)

The PPT is subdivided into n frequency with the parts chosen as equal arc divisions of the central angles of the



polyhedron.

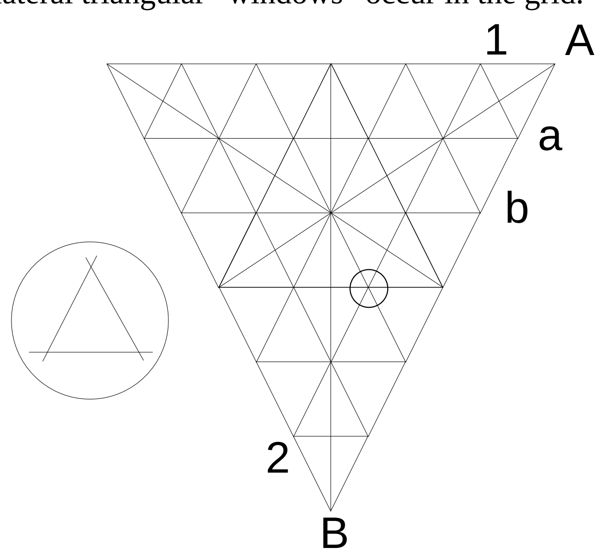
NOTE : $\overline{A1} \neq \overline{12}$

The points of subdivision on each principle side of the PPT are connected with line segments parallel to their respective sides. Each line segment intersects at a number of points which define a grid of subdivision. Due to the method of subdivision, small equilateral triangular "windows" occur in the grid.

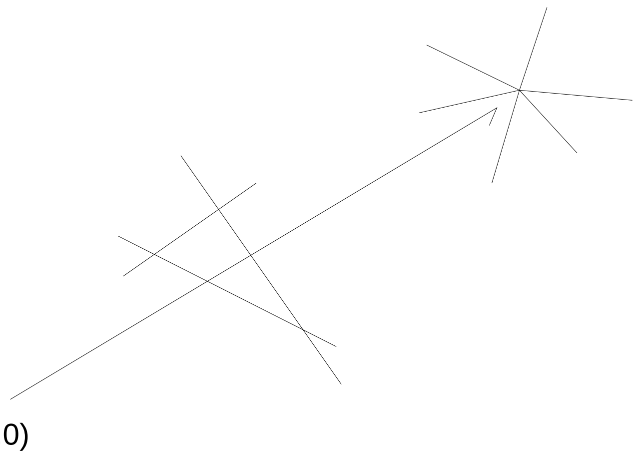
Note: AB is parallel to 12

Aa $\neq$ ab

Windows are equilateral triangles



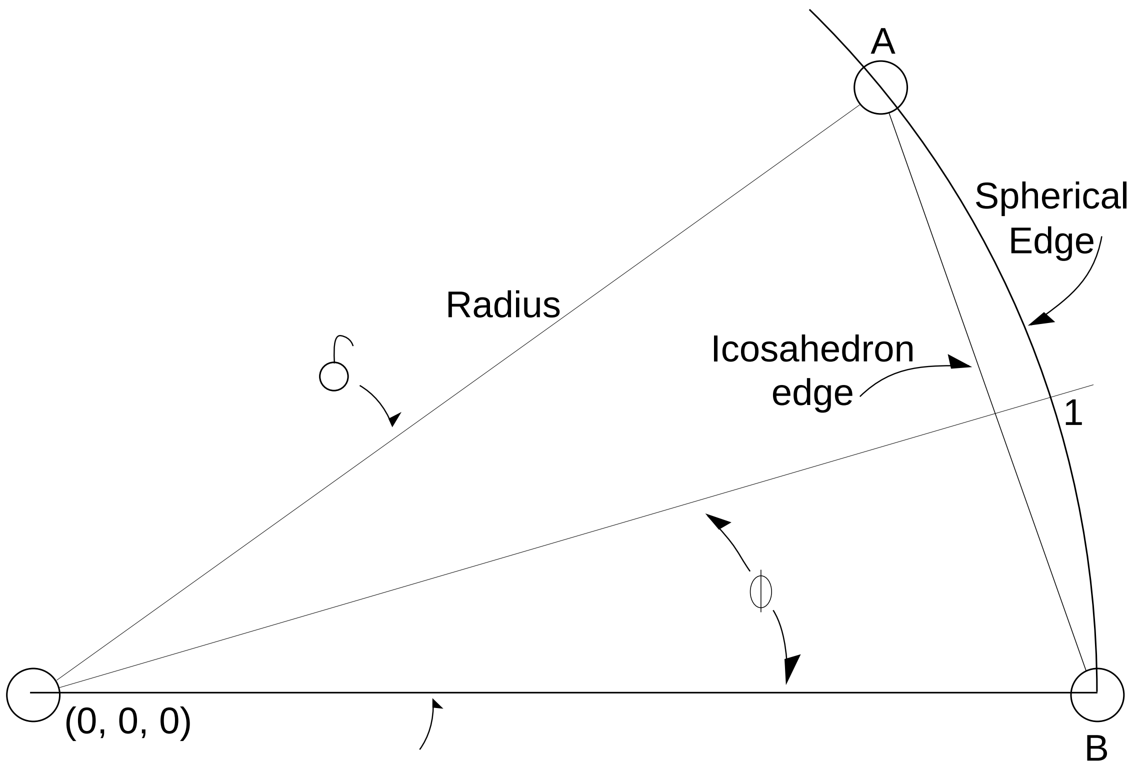
The centers of these “windows” are found on the plane of the PPT and are used as the vertices of the 3-way grid for the PPT. They are then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0, 0, 0) of the polyhedron. The elements connecting the translated vertices form the chords of a 3-way great circular grid.



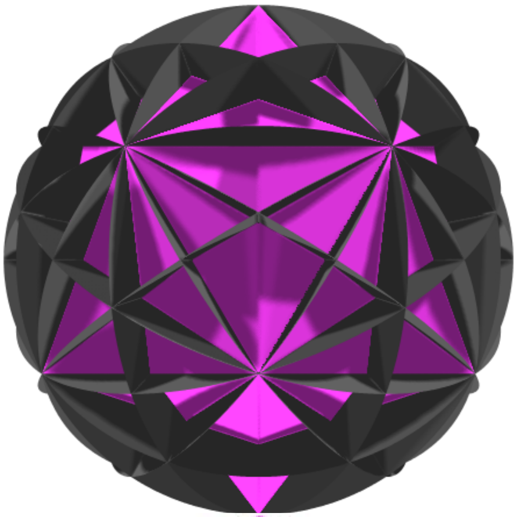
Method 3:

This method is sometimes referred to as the alternate geodesic grid. Usually, it is developed by starting with a small frequency and then subdividing further to the desired frequency by following a geometrical progression as per example

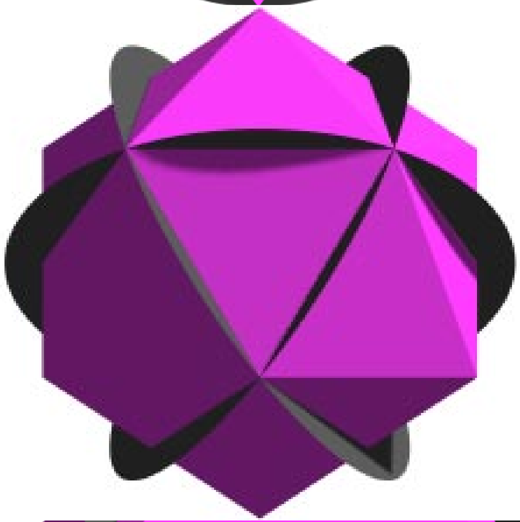
i.e., the spherical polyhedral triangle is subdivided into a low frequency subdivision,
i.e. 2 ν, with parts chosen as equal arc divisions of the central angle of the polyhedron.



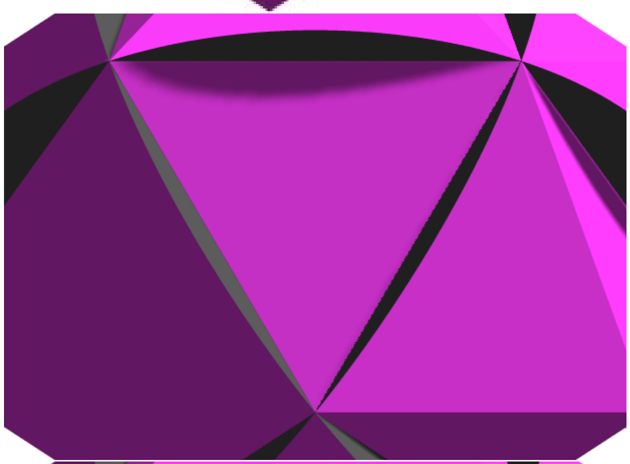
where $\phi = \dfrac{\delta}{2}$



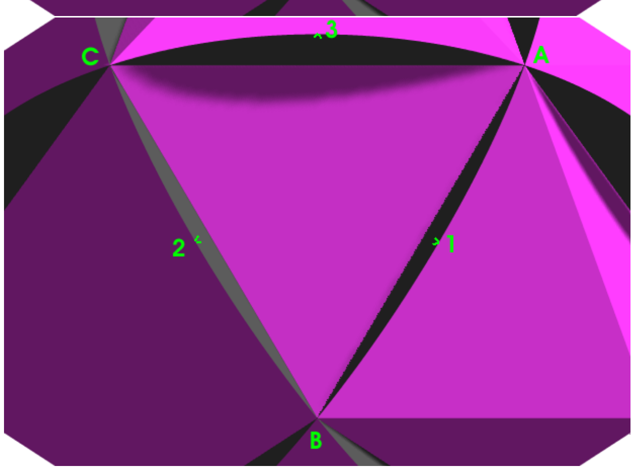
By creating great circles at every icosahedral vertex, this model is created.



By deleting the great circles at every unneeded icosahedral vertex, this model is created.

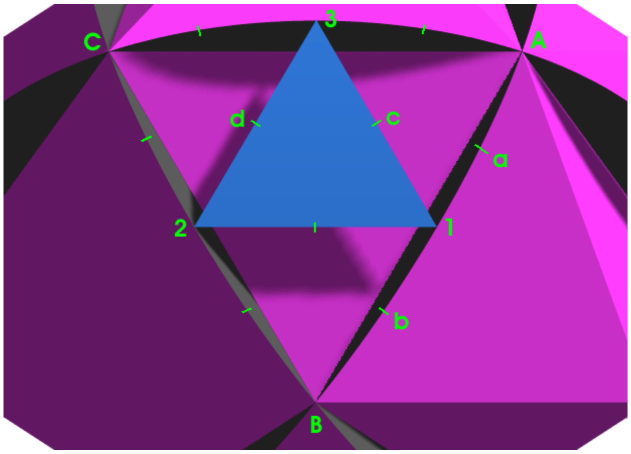


A closer view of the icosahedral face of interest.



By dividing the face using these markers, the Method 3 geometry is developed.

NOTE : $\overset{\frown}{A1} = \overset{\frown}{B1}$



By modeling a triangle on marks, 1, 2, and 3, further subdivision is created. This is noted by a, b, c, and d.

NOTE :

$$\overset{\cap}{Aa} = \overset{\cap}{a1}$$

$$\overset{\cap}{3c} = \overset{\cap}{1c}$$

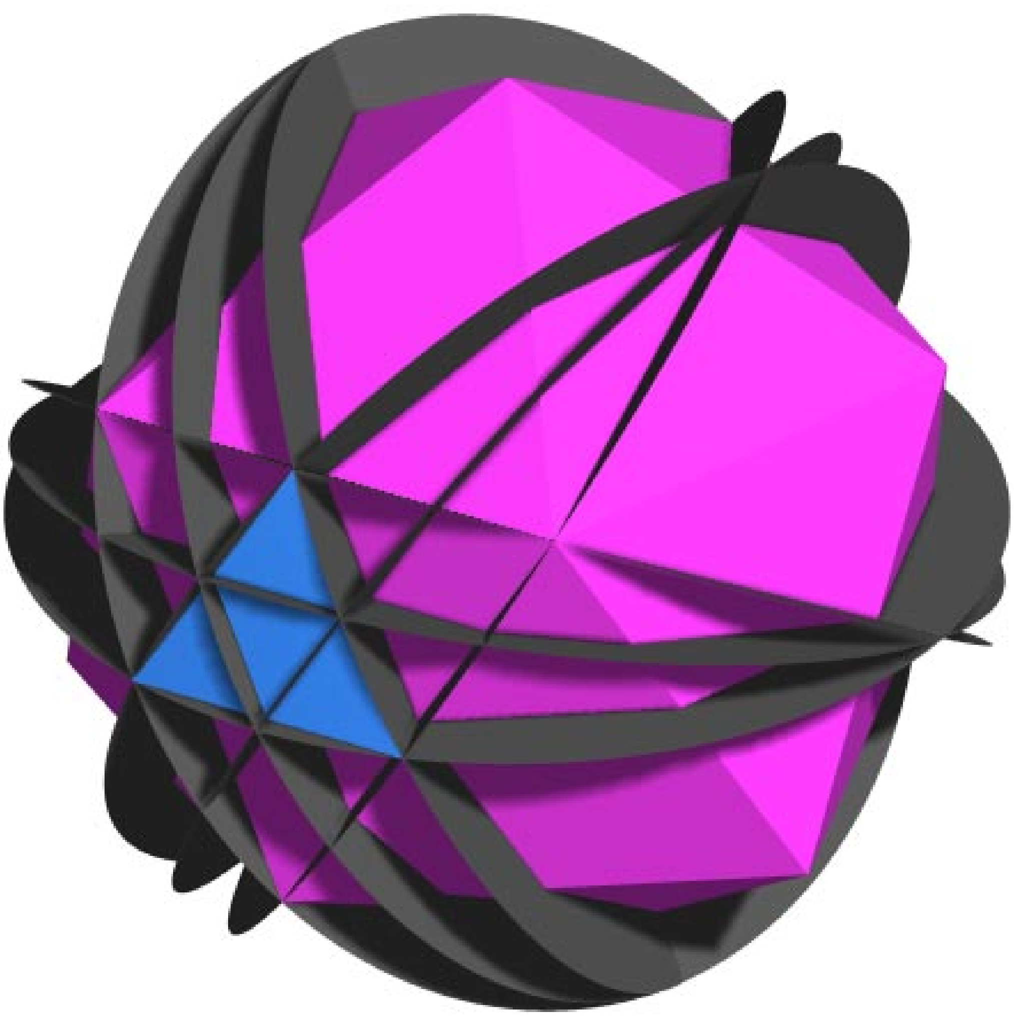
$$\overset{\cap}{1c} \neq \overset{\cap}{Aa}$$

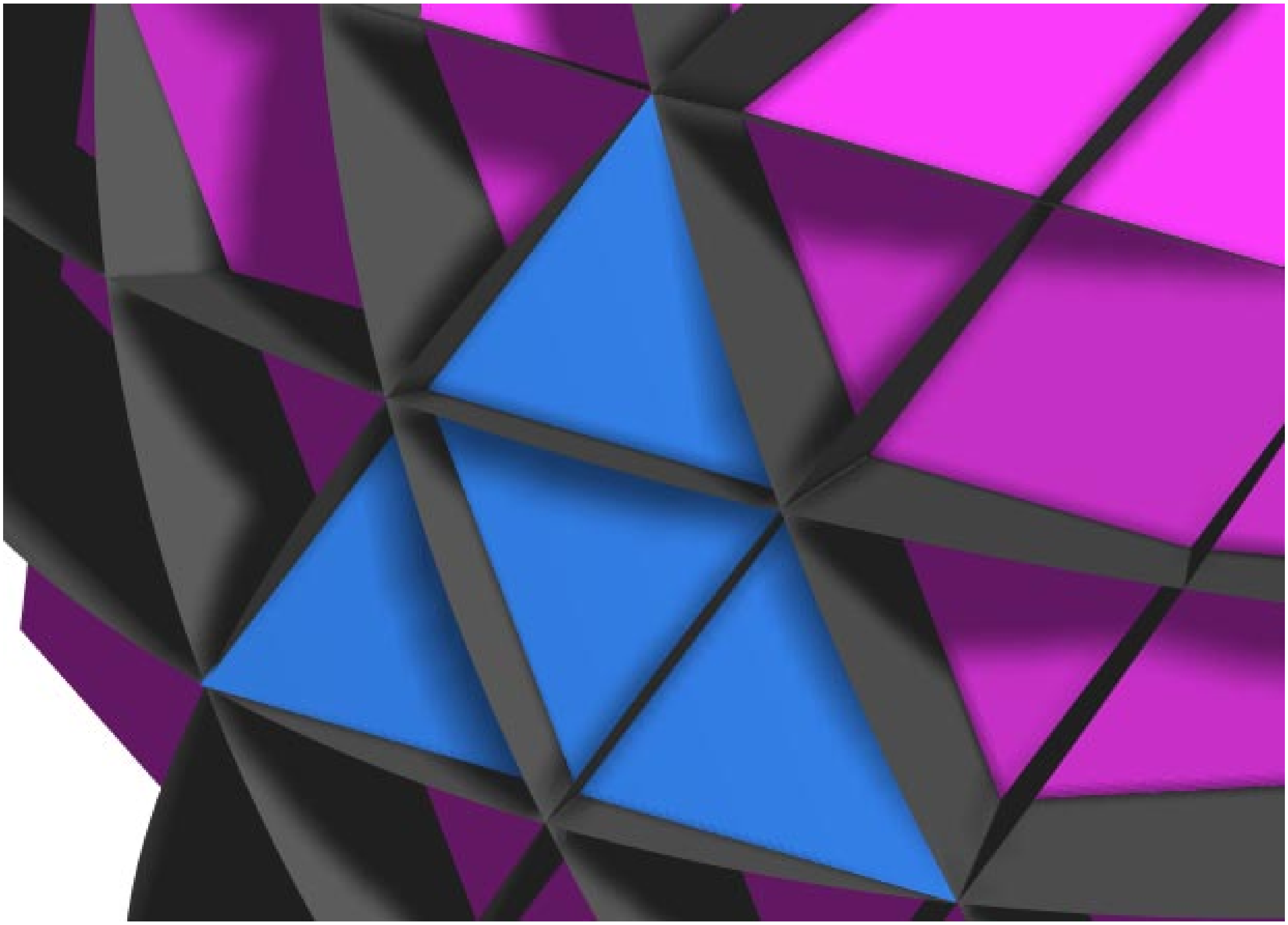
This symbol $\cap$ that appears in the above equations represents the Icosahedral Great-Circle distance marked by the points labeled in the accompanying illustrations.

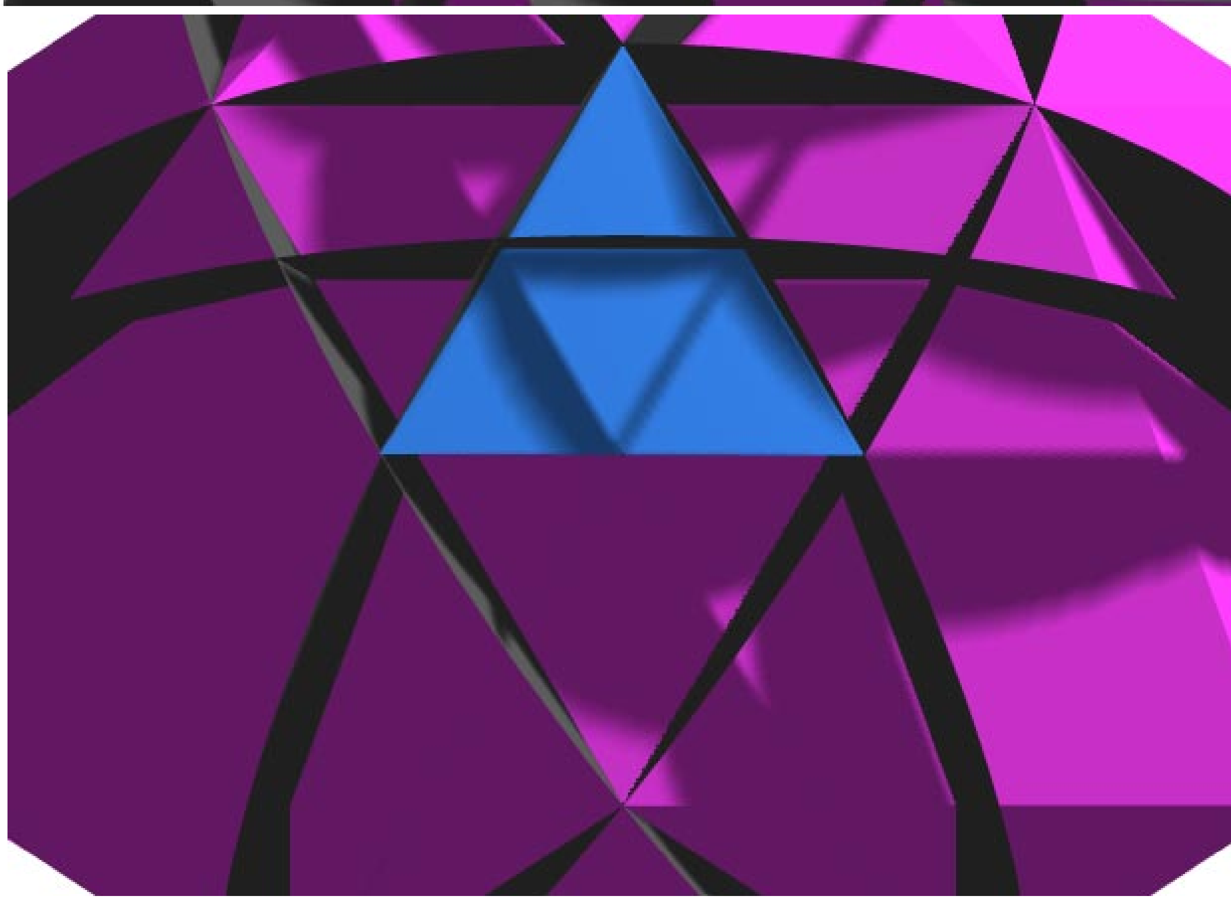
The interior points are found by passing great circle arcs through the previously found mid points of each principle side and finding the mid points of each side of each new triangle in the same manner as above.

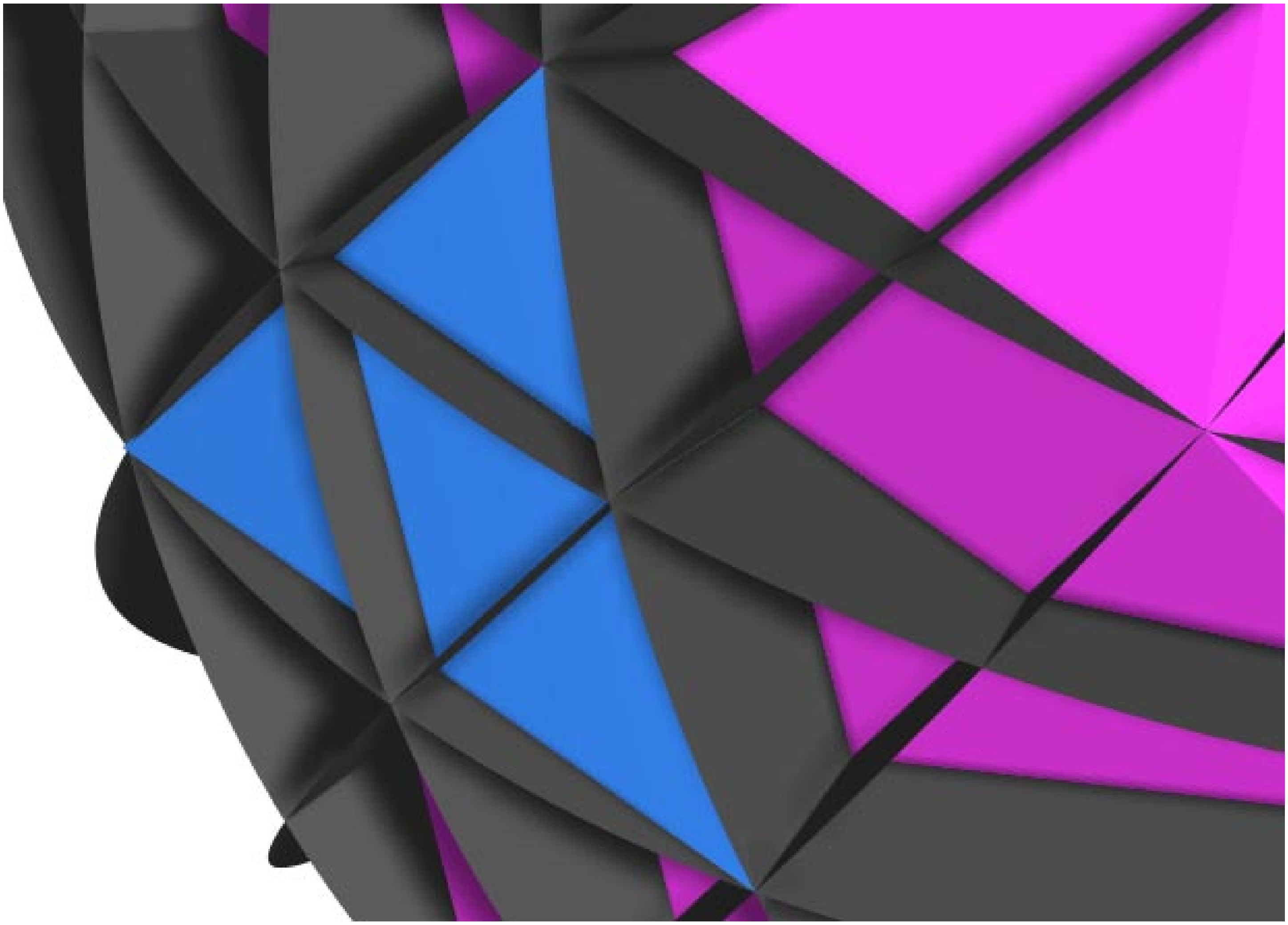
Each point is then connected with great circle arcs to complete the 3-way grid.

For further subdivisions each new triangle is subdivided as in the previous steps and connected to complete the 3-way grid.









By knowing the central angles (δ) the chord factors may be calculated by the following equation:

$$cf = 2\left(\frac{\sin \delta}{2}\right)$$

where:

cf = chord factor

δ = central angle

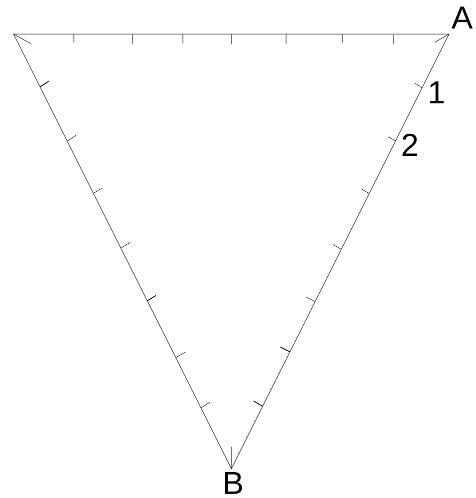
Note: The frequency increases geometrically - 2, 4, 8, 16 ...

Method 4:
This method is an alteration of methods 1-3 allowing for truncation within the equatorial zone of the spherical form. It is developed with lesser circle as well as great circle arcs so that truncation may be done without requiring special elements. A set of parallel planes, falling in the equatorial region, are provided through the geodesic sphere, perpendicular to any given polar axis. Due to the less symmetrical characteristics of this method it is used primarily for small frequency structures. The number of relative differences in edge lengths are greater than any of the other methods.

CLASS II

Method 1:

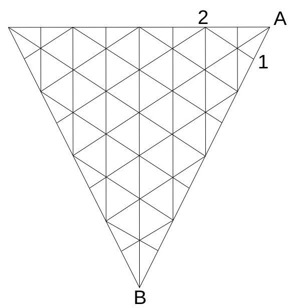
The PPT is subdivided into n frequency, with the parts chosen as equal divisions along the three principle sides.



NOTE : $\overline{A1} = \overline{12}$

Each point of subdivisions is then connected with line segments perpendicular to their respective principle side thus giving a 3-way grid comprised of equilateral and right triangles.

NOTE : $\overline{AB} \perp \overline{12}$

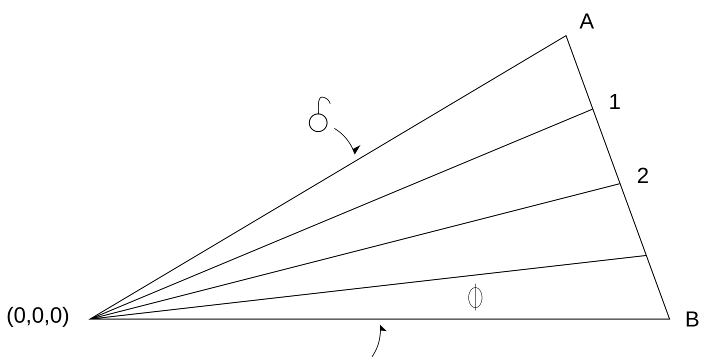


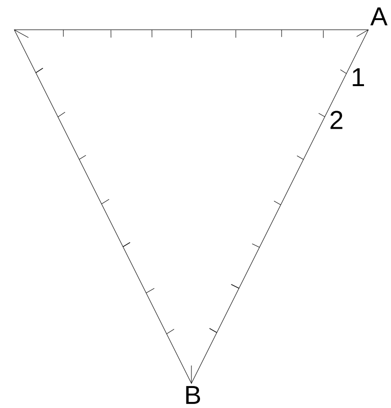
This symbol $\perp$ that appears in these equations indicates that the lines between the marked points labeled in the accompanying illustrations are perpendicular.

Each vertex on the PPT is then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0, 0, 0) of the polyhedron. The elements connecting the translated vertex form the chords of a 3-way great circular grid.

Method 2:

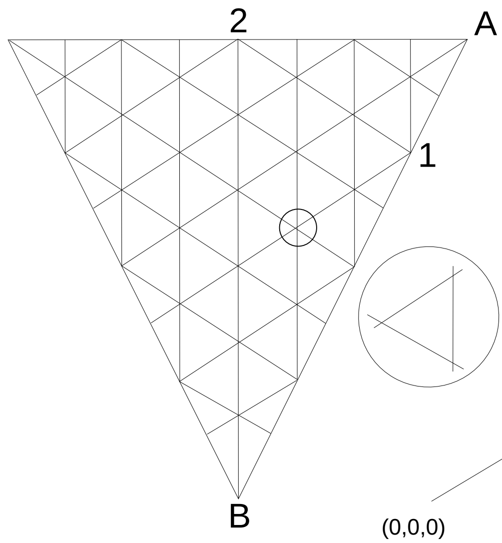
The PPT is subdivided into n frequency with the parts chosen as equal arc divisions of the central angle of the polyhedron.



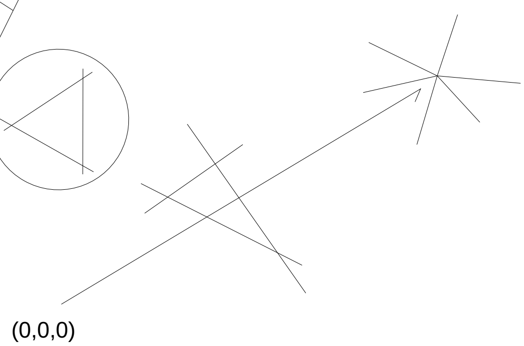


NOTE : $\overline{AB} \neq \overline{12}$

The points of subdivision on each principle side of the PPT are connected with line segments similar to method I. However, the line segments are not perpendicular to their respective sides. Upon completion of the connections a grid is created. Due to the method of subdivision, small triangular “windows” occur in the grid.



NOTE : $\overline{AB}$ not $\perp$ $\overline{12}$ Small triangular windows occur
This symbol — not $\perp$ — that appears in the above equation indicates that the lines between the marked points labeled in the accompanying illustrations are not perpendicular.

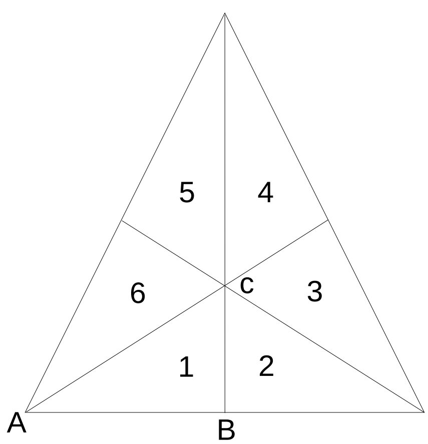


The centers of these “windows” are found and are used as the vertices of a 3-way grid for the PPT. The vertices are then translated onto the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0, 0, 0) of the polyhedron. The elements joining the translated vertices form the chords of a 3-way great circle grid.

Method 3 or triacon:

[The chord factors and other data under the name “Triacon” were developed from this method. In general, the triacon breakdown (Class II) is better for large domes because the number of different strut lengths increases arithmetically with the triacon (i.e. 6 ν has 6 different strut lengths, 8 ν has 8, 12 ν has 12 etc.) and geometrically with the alternate (Class I). For small frequency domes the difference is not that significant.]

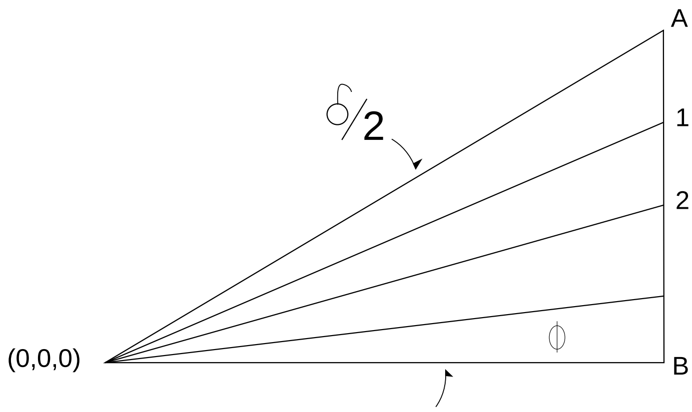
This method is sometimes referred to as the regular triacontahedral geodesic grid and was developed by Duncan Stuart.



The PPT may be described as six right triangles each being a reflection or rotation of the other.

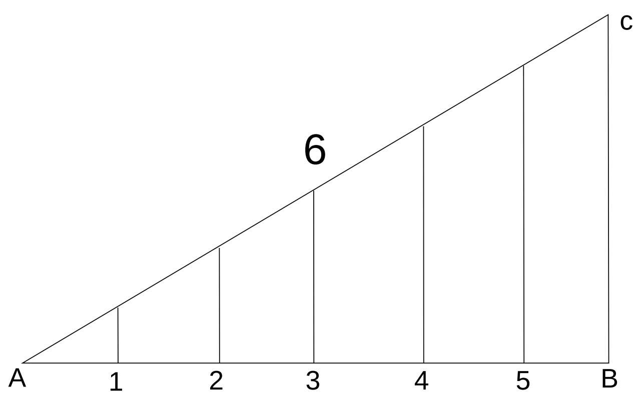
Note: ABc is a right triangle

In this method of subdivision we shall treat only triangle ABc. The remaining section of the PPT may be found through rotations and reflections of this basic unit. This is true of all methods. The line $\overline{AB}$ is into parts chosen as equal arc divisions of the central angle of the polyhedron.



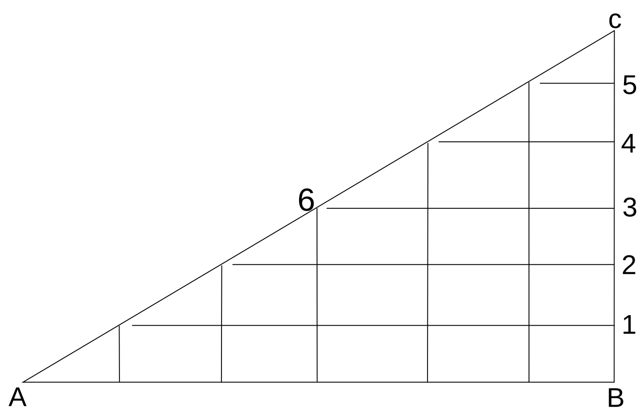
NOTE: $\overline{AB} \neq \overline{12}$

Once the subdivisions are found they are used to find the points of division on side $\overline{AC}$ and $\overline{CB}$. Perpendiculars through the points of division on side $\overline{AB}$ are extended to side $\overline{AC}$, this giving the points of subdivision on side $\overline{AC}$.



NOTE: $\overline{36} \perp \overline{AB}$

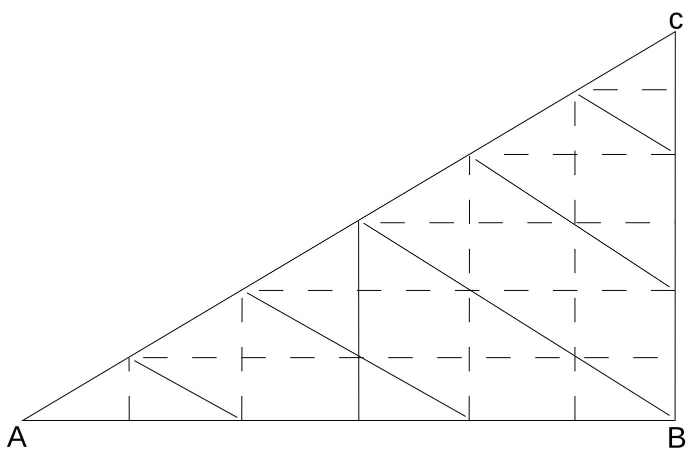
$\overline{A1} \neq \overline{23}$



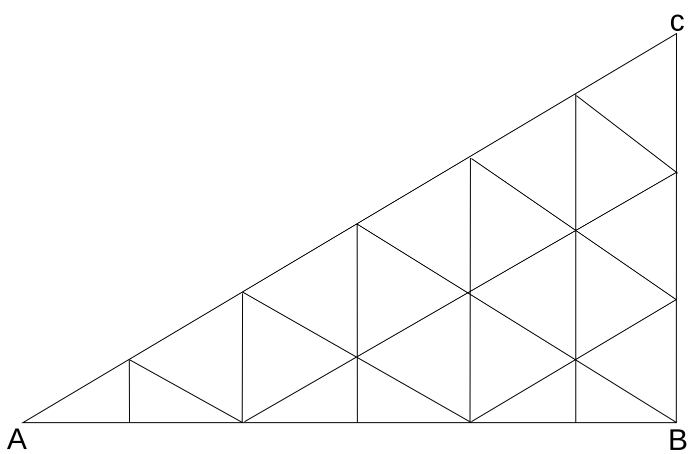
The points of division on the side $\overline{CB}$ were formed by extending a line through the points of subdivision on side $\overline{AC}$ perpendicular to side $\overline{CB}$.

NOTE: $\overline{36} \perp \overline{CB}$

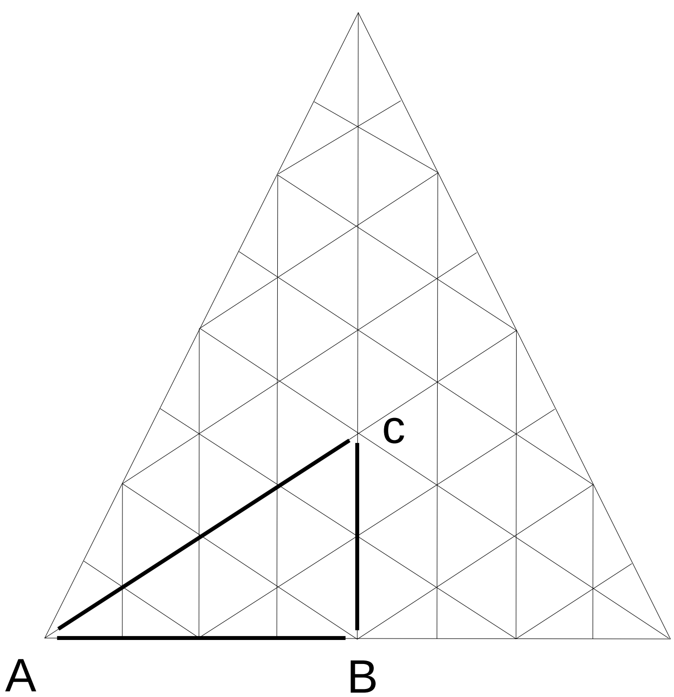
$\overline{12} \neq \overline{34}$



Having acquired the points of subdivision along the three sides of the triangle, diagonals are drawn from each point on side $\overline{AC}$ to alternate points of sides $\overline{AB}$ and $\overline{BC}$.



To complete the 3-way grid connect alternate points of subdivision of side $\overline{AB}$ to the points of division of side $\overline{BC}$.



Through rotations and reflections of the, basic unit and its subdivisions, the entire 3-way gridding of the PPT may be found.

The vertices of the 3-way grid are then translated to the surface of the circumscribed sphere along a line passing through the respective vertex and the origin (0, 0, 0) of the polyhedron. The elements joining the translated vertices form the chords of a 3-way great circle grid.

Method 4:

[This method is basically the same as method 3 except that instead of dividing side $\overline{AB}$ of the right triangle with the equal arc divisions, side $\overline{AC}$ is divided. The rest of the procedure is the same, given the new starting point.]

All Artwork, Graphics and Illustrations were created or made by Jay Salsburg, Design Scientist, <jay@salsburg.com>